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A METHOD OF DERIVING EULER'S EQUATION IN THE CALCULUS OF VARIATIONS.

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The derivation of Euler's equation can be made by means of integrals of the form

$$(1) \qquad \int [A(x, y) + B(x, y)y']\,dx, \quad y' = \frac{dy}{dx},$$

which are independent of the path of integration. Invariant integrals of this type play an important role in the proofs that the usual conditions in the calculus of variations are sufficient to insure a minimum or a maximum, and their introduction in connection with the derivation of Euler's equation makes it possible to simplify considerably the presentation of the whole theory.

In the first section below there is a simple discussion of the conditions under which an integral of the form (1) is independent of the path, and in the second section these results are applied to the derivation of Euler's equation.

§1. INVARIANT INTEGRALS.

In the integral (1) suppose that the functions $A(x, y)$ and $B(x, y)$ are continuous in a certain region R of the xy-plane. Along an arc C_{12}

$$(2) \qquad y = y(x), \quad x_1 \leqq x \leqq x_2,$$

which joins two given points (x_1, y_1) and (x_2, y_2), lies in R, and for which the function $y(x)$ is continuous and has a continuous derivative, the integral I will have a value

$$(3) \qquad I=\int_{x_0}^{x_1}\{A[x,\ y(x)]+B[x,\ y(x)\ y'(x)]\}dx$$

denoted by I_{12} or $I(C_{12})$. If for two arbitrarily chosen points $(x_1,\ y_1)$, $(x_2,\ y_2)$, the values I_{12} are all the same however the arc C_{12} is chosen, the integral I is said to be indepenent of the path.

The sum of the values of such an invariant integral taken along the sides of a triangle of arcs of the type (2) is zero. To prove this, let the vertices of the triangle be denoted by 1, 2, 3, and the sides by C_{23}, C_{31}, C_{12}. It is always possible to pass an arc D of the type (2) through the three points 1, 2, 3, so that for a triangle such as the one in the figure,

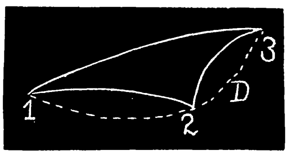

Fig. 1.

$$I(C_{12})+I(C_{23})=I(D_{12})+I(D_{23})$$
$$=I(D_{13})=I(C_{13}).$$

But for any arc C_{13},

$$(4) \qquad I_{13}=-I_{31},$$

since I_{31} is found from an integral of the form (3) by simply changing the limits. Hence along the sides of the triangle

$$I_{12}+I_{23}+I_{31}=0.$$

A similar theorem holds for any polygon of arcs C_{12}, C_{23}, ..., $C_{n-1,1}$, of the type (2). For select a point 0 not on any of the ordinates of the vertices of the polygon, and join it to them by straight lines. Then

$$\begin{aligned} &I_{01}+I_{12}+I_{20}=0,\\ &I_{02}+I_{23}+I_{30}=0,\\ &\cdot \quad \cdot \quad \cdot \quad \cdot\\ &I_{0,n-1}+I_{n-1,1}+I_{10}=0, \end{aligned}$$

and by adding and using equations similar to (4),

$$I_{01}+I_{12}+\ldots+I_{n-1,1}=0.$$

The integral

$$\phi(x,\ y)=\int_{(x_0,\ y_0)}^{(x,\ y)}(A+By')dx$$

taken from the fixed point (x_0, y_0) to the point (x, y) over any continuous curve formed of a finite number of arcs of the type (2) defines a single-valued function $\phi(x, y)$. For any two broken curves joining (x_0, y_0) with (x, y) form a polygon over which the value of the integral is zero, and the two values found by integrating from (x_0, y_0) to (x, y) over the two broken curves, are equal on account of the property (4).

The difference of two values of $\phi(x, y)$ corresponding to points on the same ordinate has the value

$$\phi(x, y) - \phi(x, y_1) = \int_{y_1}^{y} B(x, y)\,dy. \tag{5}$$

This difference can in fact be written in the form

$$\phi(x, y) - \phi(x, y_1) = \int_{(x-h,\, y_1)}^{(x,\, y)} (A + By')\,dx - \int_{x-h}^{x} A(x, y_1)\,dx, \tag{6}$$

since from the figure it is evident that

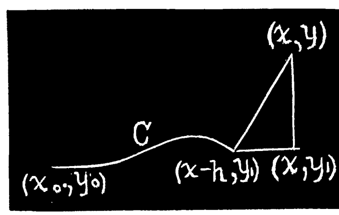

Fig. 2.

$$\phi(x, y) = I(C) + \int_{(x-h,\, y_1)}^{(x,\, y)} (A + By')\,dx,$$

$$\phi(x, y_1) = I(C) + \int_{x-h}^{x} A(x, y_1)\,dx.$$

Equation (6) may also be written, by a change of variable,

$$\phi(x, y) - \phi(x, y_1) = \int_{(x-h,\, y_1)}^{(x,\, y)} A\,dx + \int_{y_1}^{y} B\,dy - \int_{x-h}^{x} A(x, y_1)\,dx.$$

When h approaches zero the first and third of these integrals vanish, and the second approaches the value given in equation (5).

The derivatiyes of $\phi(x, y)$ can now be readily calculated. From (5) it follows that

$$\frac{\partial \phi}{\partial y} = B(x, y).$$

It is evident from a figure similar to figure 2 that

$$\phi(x, y) - \phi(x_1, y) = \int_{x_1}^{x} A(x, y)\,dx,$$

so that

$$\frac{\partial \phi}{\partial x}=A(x, y).$$

A necessary and sufficient condition for the integral I to be independent of the path in a region R is therefore that a single-valued function $\phi(x, y)$ exists in R having the derivatives

$$\frac{\partial \phi}{\partial x}=A, \quad \frac{\partial \phi}{\partial y}=B. \tag{7}$$

The sufficiency of this condition was not proved above, but follows easily with the help of the fundamental theorem of the integral calculus. For along any arc (2)

$$\int_{(x_1, y_1)}^{(x_2, y_2)}(A+By')\,dx=\int_{(x_1, y_1)}^{(x_2, y_2)}\left(\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}y'\right)dx=\phi(x_2, y_2)-\phi(x_1, y_1).$$

Another criterion for the invariancy of I is the following:

If the functions A and B in the integral (1) *have continuous partial derivatives* $\frac{\partial A}{\partial y}$ *and* $\frac{\partial B}{\partial x}$ *in a simply connected region* R^*, *then a necessary and sufficient condition for the integral I to be independent of the path, is that the equation*

$$\frac{\partial A}{\partial y}=\frac{\partial B}{\partial x} \tag{8}$$

is identically satisfied in R.

The condition is necessary, for from the previous theorem a function ϕ must exist with the derivatives (7), and we have

$$\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x}=\frac{\partial^2 \phi}{\partial x \partial y}-\frac{\partial^2 \phi}{\partial y \partial x}=0.$$

To prove the sufficiency, suppose first that R is a rectangle with one corner at (x_0, y_0). Then the values of I taken from (x_0, y_0) to (x, y) along lines parallel to the x and y axes define a function

$$\phi(x, y)=\int_{x_0}^{x}A(x, y_0)\,dx+\int_{y_0}^{y}B(x, y)\,dy, \tag{9}$$

*R is said to be simply connected if any two of its points can be connected by a continuous curve, and if the interior of any continuous, closed, non-intersecting curve in R is also entirely within the region.

which is readily seen with the help of equation (8) to have the partial derivatives A and B. Consequently in any rectangle where condition (8) is satisfied, the integral I is independent of the path.

In a more general simply-connected region R, the value of I taken around any polygon whose sides are parallel to the x- or y-axis is zero. For by continuing all the sides of the polygon, its interior is divided into rectangles, and the sum of the values of I taken around these rectangles in the positive direction* is the value of I taken in the positive direction about the original polygon. This follows because we have to integrate along a side of any rectangle twice, in opposite directions, unless the side is an edge of the original polygon. But since R is simply connected, the rectangles are all in R when condition (8) holds, and the value of I taken around the edge of such a rectangle is zero. Hence if we integrate from a fixed point (x_0, y_0) to the point (x, y) in R along a broken curve consisting of straight lines parallel to one or the other of the axes, a single-valued function $\phi(x, y)$ is defined. This function like the function defined by equation (9) has A and B for its partial derivatives.

For the application to be made in the next section, the following remark is important.

If the integral (1) *takes the same value over all continuous curves consisting of a finite number of arcs of the type* (2) *joining two fixed points* (x_1, y_1) *and* (x_2, y_2), *then it must be independent of the path in the same way for any two points in the strip of the plane between the ordinates* $x=x_1$ *and* $x=x_2$.

Let 3 and 4 be any two points in this strip, and join 1 with 3, and 4 with 2 by fixed arcs of the type (2). Then however the points 3 and 4 are connected by broken arcs, we will always, by hypothesis, have the same value for

$$I_{13}+I_{34}+I_{42}.$$

Consequently, I_{34} is independent of the path also.

§3. The Derivation of Euler's Equation.

The problem of the calculus of variations which we shall consider here is the problem of finding a curve which joins two given fixed points (x_0, y_0) and (x_1, y_1), and gives a maximum or a minimum value to an integral of the form

$$J=\int f(x, y, y')\,dx.$$

The function f under the integral sign will be supposed to have continuous derivatives of the first and second orders for points (x, y) in a region R of

*I. e. keeping the interior of the rectangle on the left.

the xy-plane and all values of y'. Evidently J will then have a well-defined value over any continuous curve consisting of a finite number of arcs of the type (2) in R.

Suppose that one of these arcs

$$(10) \qquad E; \qquad y=y(x), \quad x_0 \leqq x \leqq x_1,$$

joins the points 0 and 1 in R and gives J a minimum value. The family of arcs

$$v; \qquad \bar{y}=y(x)+a\eta(x), \quad x_0 \leqq x \leqq x_1,$$

where a is an arbitrary constant and $\eta(x)$ satisfies the conditions

$$\eta(x_0)=\eta(x_1)=0,$$

also joins the points 0 and 1 and include E for the value $a=0$. If the function $\eta(x)$ has a derivative which is continuous except perhaps at a finite number of points in the interval $x_0 \leqq x \leqq x_1$, then along any arc v the integral J will have a value

$$J(a)=\int_{x_0}^{x_1} f(x, \bar{y}, \bar{y}')\,dx,$$

which is a function of a. Since v reduces to E when $a=0$, it follows that for $a=0$ the function $J(a)$ must have a minimum and $\frac{dJ}{da}$ must be zero.

The derivative $\frac{dI}{da}$ for $a=0$ is easily found to be

$$(11) \qquad \left(\frac{dJ}{da}\right)_{x=0}=\int_{x_0}^{x_1}\left(\frac{\partial f}{\partial y}\eta+\frac{\partial f}{\partial y'}\eta'\right)dx,$$

where in the derivatives of f the values of y and y' along E are substituted. This derivative must vanish however the function $\eta(x)$ satisfying the conditions given above, is chosen. We see then that in the $x\,\eta$-plane the integral (11) is independent of the path for all curves joining the points $(x_0, 0)$ and $(x_1, 0)$, and consequently independent of the path anywhere in the strip of the $x\,\eta$-plane between the ordinates $x=x_0$ and $x=x_1$. There must therefore exist a function $\phi(x, \eta)$ which has the derivatives

$$(12) \qquad \frac{\partial\phi}{\partial x}=\frac{\partial f}{\partial y}\eta, \quad \frac{\partial\phi}{\partial\eta}=\frac{\partial}{\partial y'}.$$

From the first of these equations,

$$\phi=\eta\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+H(\eta), \tag{13}$$

and from the second,

$$\frac{\partial f}{\partial y'}=\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+H', \tag{14}$$

an equation which must be true at every point of E. It is evident by deriving again for η, that H'' must be zero, and H has the form

$$H=c\,\eta+d. \tag{15}$$

If the curve E has corners for the values $x=\alpha, \alpha_2, \ldots, \alpha_n$ in the interval $x_0\leqq x\leqq x_1$, then the functions (12) may have discontinuities, but they will certainly be continuous in any strip of the $x\,\eta$-plane between two ordinates $x=\alpha_k$, $x=\alpha_{k+1}$. In such a strip ϕ must therefore be continuous and it is evident from equations (13) and (15) that c has the same value throughout the strip.

In two different strips c is also the same. For simplicity consider the case when there is but one value $x=\alpha$, and let

$$\phi=\eta\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+c\,\eta+d,\qquad \psi=\eta\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+e\,\eta+f \tag{16}$$

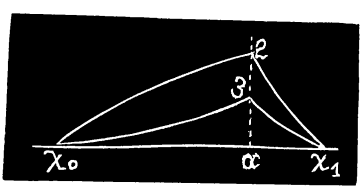

Fig. 3.

be the two functions (13) in the two strips. Along two curves such as those in figure 3,

$$I=\phi_2-\phi_0+\psi_1-\psi_0=\phi_3-\phi_0+\phi_1-\phi_3,$$

the subscripts indicating the points at which the values of the functions are to be taken. Hence

$$\phi_2-\phi_3=\psi_2-\psi_3,$$

from which it follows with the help of equations (16) that $c=e$.

From equation (14), therefore,

$$\frac{\partial f}{\partial y'}=\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+c, \tag{17}$$

where c has the same value throughout the whole interval $x_0 \leqq x \leqq x_1$.

Several important conclusions can be drawn from equation (17). In the first place consider a point 2 on E when y' is continuous. The left member of (17) may be regarded for the moment as a function of the two varibles y' and x, the latter entering explicitly and also in the function $y(x)$. Then the values x_2, y_2, y'_2 at the point 2 on E furnish a solution of equation (17). If $\frac{\partial^2 f}{\partial y'^2}$ is different from zero for these values, then the theory of implicit functions tells us that equation (17) has but one continuous solution $y'(x)$ which reduces to y'^2 when $x = x_2$, and this solution has a continuous derivative. The values of $y'(x)$ on the curve E near the point 2 constitute this solution, and by differentiating equation (17) for x we derive the following theorem:

If an arc E

$$y = y(x), \quad x_0 \leqq x \leqq x_1,$$

minimizes the integral J, then at any point on E where $y'(x)$ is continuous and $\frac{\partial^2 f}{\partial y'^2}$ different from zero, the function $y(x)$ must also have a second derivative and satisfy the Euler differential equation

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0.$$

It is possible to show further from equation (17) that a minimizing arc can not in general have corner points. At a point (x_2, y_2) of E where $\frac{\partial^2 f}{\partial y'^2}$ is different from zero for all values of y', it is evident that the first member of equation (17) can equal the second for at most one value of y', since $\frac{\partial f}{\partial y'}$ is monotomic. Hence it would be impossible for y' to be discontinuous at (x_2, y_2).

An arc E which minimizes the integral J can not have a corner point at any point (x_2, y_2) where $\frac{\partial^2 f}{\partial y'^2}$, is different from zero for all values of y'. If $\frac{\partial^2 f}{\partial y'^2}$ is different from zero at any point in R for all y''s, then no minimizing curve whatsoever with corner points is possible anywhere in R.